Numerical Solution of Large-Scale Inverse Problems

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SAMBa SLS
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Introducing Inverse Problems

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When solving a

DIRECT PROBLEM

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- **Direct problems**: from the cause of a observed phenomenon, to its effect.
- **Inverse problems**: from the effect of an observed phenomenon, to its cause.

Some examples

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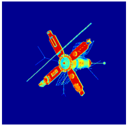
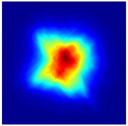
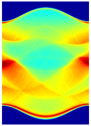
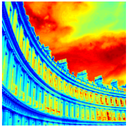
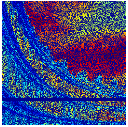
model	input	output
image deblurring (astronomical imaging)		
PSF, convolution		
computerised tomography (industrial, medical)		
X-rays, line integrals		
matrix completion		
		

Inverse problems are ill-posed

According to Hadamard (1923), a problem is **ill-posed** if

- (the) solution is not unique

or

- (the) solution does not depend continuously on the given data

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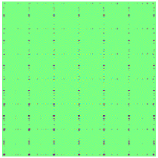
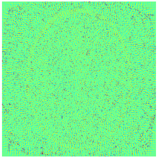
Indeed, this is what happens:

deblurring



CT



Understanding what goes wrong

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Consider the SVD of $A \in \mathbb{R}^{N \times N}$, for these examples $N = 65536$:

$$A = U \Sigma V^T$$

When there is noise, $b = b^{ex} + e$ (for this example $\|e\|_2 / \|b\|_2 = 10^{-2}$):

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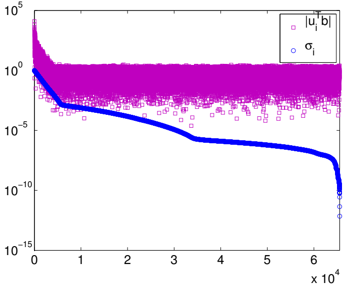

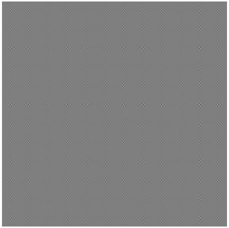

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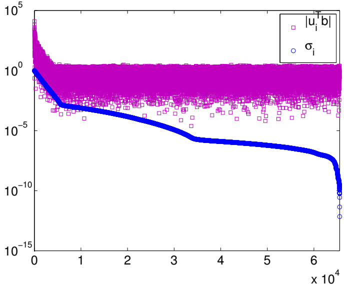

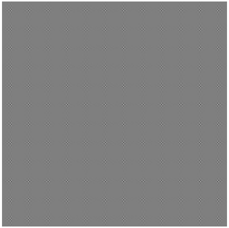

Applying direct regularisation

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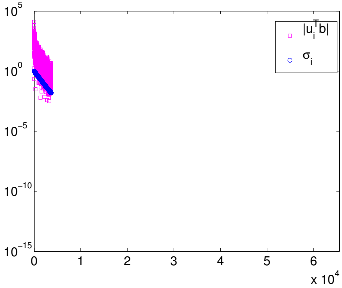
For this example: $k = 3653$.



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Tikhonov regularization:

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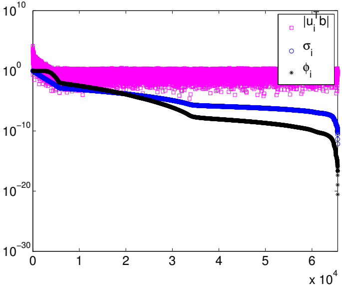
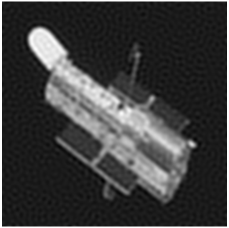
For this example: $\lambda = 1.76 \cdot 10^{-4}$, $L = I$.





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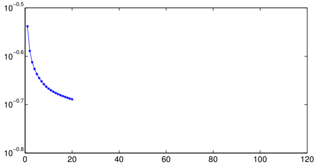
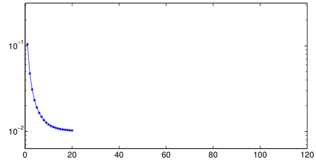


relative residuals



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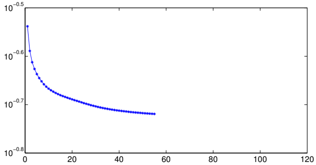

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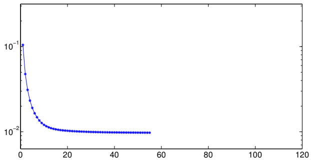

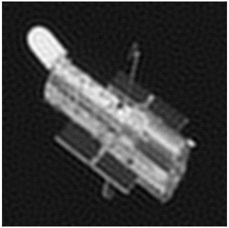

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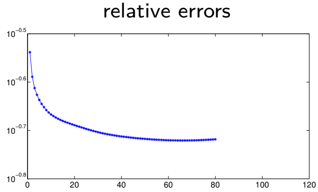

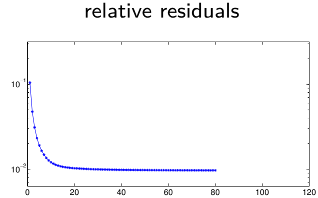

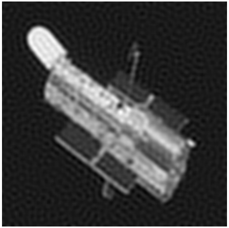

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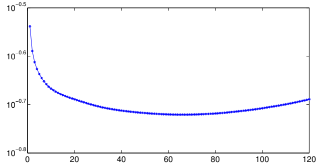

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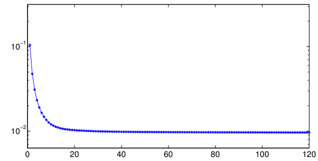
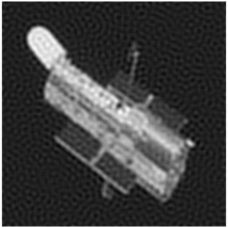
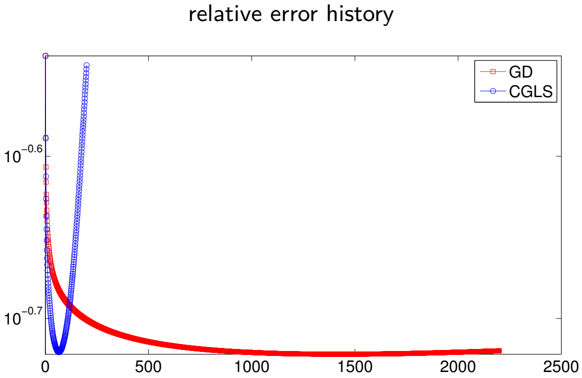




Gradient Descent approach VS. Krylov Subspaces approach



Enforcing sparsity

- By penalisation:

$$x_\lambda = \arg\min_{x \in \mathbb{R}^N} \|Ax - b\|_2^2 + \lambda \|x\|_1$$

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- By imposing constraints:

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x_+ &= \arg\min_{x\geq 0} \|Ax - b\|_2
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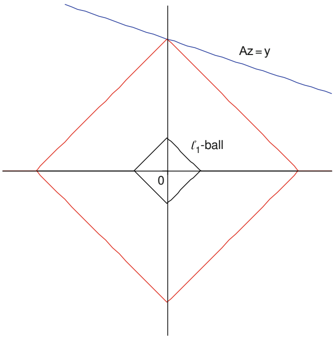

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[Image courtesy: Fornasier and Rauhut. *Compressive Sensing*, 2011]

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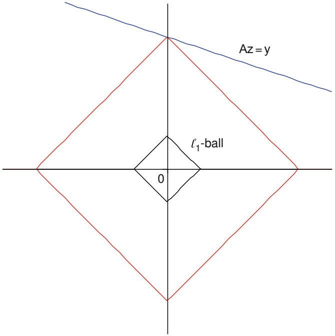

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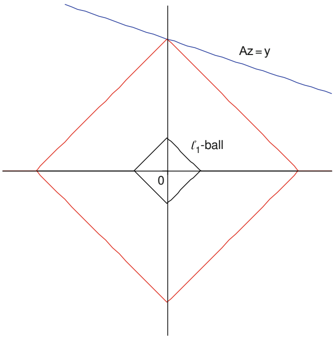

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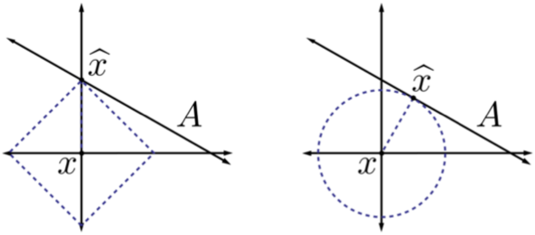
the sparsest solution is recovered!

$\|\cdot\|_1$ vs. $\|\cdot\|_2$ minimisation



[Image courtesy: Baraniuk et al. *An Introduction to Compressive Sensing*, 2013]

Examples of sparsity transforms Ψ

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original

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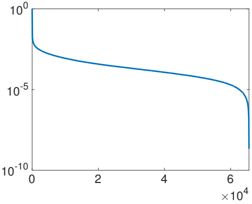
2D dct

dct coefficients

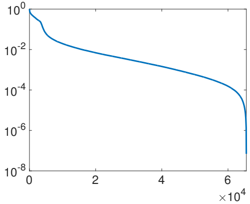


Examples of sparsity transforms Ψ

original

2D wavelets

wavelet coefficients

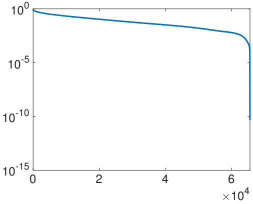


Examples of sparsity transforms Ψ

original

gradient

gradient coefficients



Incomplete information & compressive sensing theory

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- **sparsity** (Ψx is k-sparse)
- **randomness** (the rows of A are chosen uniformly at random)
- "**incoherence**" ((A, Ψ) with "coherence" $\mu \ll \sqrt{N}$)

and

$$M \geq C \cdot \mu^2 \cdot (k \cdot \log(N)),$$

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and

$$M \geq C \cdot \mu^2 \cdot (k \cdot \log(N)),$$

then the **compressive sensing** theory guarantees that we can recover $x \in \mathbb{R}^N$ with overwhelming probability by solving

$$\min_{x \in \mathbb{R}^N} \|\Psi x\|_1 \quad \text{s.t.} \quad \|Ax - b\|_2 \leq \varepsilon.$$

Making sense of "(in)coherence"

Let a^i, $i = 1, \ldots, N$, ψ^j, $j = 1, \ldots, N$ be two basis of $\mathbb{R}^N$. Then, the **coherence** μ between a^i and ψ^j is defined as

$$\mu = \sqrt{N} \max_{1 \leq i, j \leq N} \frac{|\langle a^i, \psi^j \rangle|}{\|a^i\|_2 \|\psi^j\|_2} .$$

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Note that

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[time for a movie...]

On the whiteboard...

J. F. Cai, E. J. Candes, and Z. Shen

A SINGULAR VALUE THRESHOLDING ALGORITHM FOR MATRIX COMPLETION

SIAM J. Optim., Vol 20, No 4, pp. 1956–1982

http://epubs.siam.org/doi/ref/10.1137/080738970